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Solution of a Linearized Kinetic Model for an Ultrarelativistic Gas

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A linearized model of the Boltzmann equation for a relativistic gas is shown to be reducible, in the ultrarelativistic limit and for $(1+1)$-dimensional problems, to a system of three uncoupled transport equations, one of which is well known. A general method for solving these equations is recalled, with a few new details, and applied to the solution of two boundary value problems. The first of these describes the propagation of an impulsive change in a half space and is shown to give an explicit example of the recently proved result that no signal can propagate with speed larger than the speed of light, according to the relativistic Boltzmann equation. The second problem deals with steady oscillations in a half space and illustrates the meaning of certain recent results concerning the dispersion relation for linear waves in relativistic gas.

KEY WORDS: Relativistic Boltzmann equation; transport equations.

1. INTRODUCTION

In a recent paper[1] a proof was given that infinitesimal disturbances, traveling in a gas otherwise in equilibrium, propagate, according to the relativistic Boltzmann equation, at a speed less than the speed of light c. Further research was devoted to the study of the dispersion relation, i.e., the relation between frequency and wave number according to relativistic theory.[2,4] For this purpose relativistic kinetic models were introduced. In Ref. 3 the model proposed in Ref. 2, which appeared to be the most direct analog of the classical BGK model,[5,7] was shown to have a peculiar behavior, in that the discrete spectrum was imbedded in a continuous spectrum in the low-frequency limit. This led[3] to the conjecture that a con-

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stant value of the collision frequency is not a reasonable equivalent of the analogous assumption for the traditional BGK model. Accordingly it was suggested that a value of that frequency proportional to p^0 (in the reference frame of the unperturbed gas) should be taken, if p^0 denotes the time component of the molecule momentum. After submitting the present paper, a referee pointed out to the author that the same model had been proposed much earlier by Anderson and Witting,[8] who applied it to the computation of transport coefficients.

The model was studied in detail in Ref. 4 where the phase speeds and attenuation rates of thermal, sound, and shear waves were computed as functions of the frequency. The results were in agreement with the expectations.

In particular, the dispersion relation for each kind of wave turned out to have solution if and only if the frequency was less than a critical value of ω_c (of the order of the collision frequency). This circumstance was already present in the classical case,[9,11,6,7] where Sirovich and Thurber suggested to look for the roots of the analytically continued dispersion relation and the present author gave a meaning to this procedure.[11,6,7] In the relativistic case, however, the analytically continued dispersion relation leads to phase speeds larger than c, thus violating the basic result of Ref. 1.

Since the analytically continued dispersion relation has only an approximate meaning and is significant only in the neighborhood of the critical frequency ω_c, it appears of interest to investigate the speed of propagation of disturbances in a boundary value problem. For this purpose, one considers a gas occupying a half space, whose boundary undergoes an oscillation of either mechanical or thermal nature and looks for the corresponding solution of a linearized kinetic model. In the case of a classical gas, this kind of problem can be solved analitically for both shear[11] and sound[12] waves. The procedure is much more complicated in the second case and this explains why the problem had been previously treated in a partly numerical manner. The procedure of Ref. 12 could be extended, in principle, to thermal waves although this does not appear to have been done.

The treatment of relativistic waves propagating in a gas occupying a half space appears to be extremely complicated in the general case. It is feasible, however, for a high temperature gas, i.e., in the so-called ultrarelativistic limit, as will be shown in this paper. A particularly appealing feature of the results of this paper is that the study of one-dimensional problems connected with the kinetic model studied in Ref. 4 is reduced to transport equations well known in radiative transfer[12] and neutron transport.[13] In Section 2 this reduction will be performed, while Section 3 is devoted to recalling certain results on the simplest transport equation obtained in the previous section. Analogous results for the other equations are indicated in Section 6. Applications to initial value problems and propagation of oscillations follow.

2. THE TRANSPORT EQUATIONS IN THE ULTRARELATIVISTIC LIMIT

The kinetic model to be employed to describe the time evolution of the molecular distribution function is a relativistic version of a velocity dependent Krook model.[15,16,6,7] This model was proposed in Refs. 3 and 4, but, as mentioned in the previous section, had been introduced much earlier by Anderson and Witting[8] to illustrate the computation of relativistic transport coefficients. The basic equation is

$$p^{\alpha} \frac{\partial f}{\partial x^{\alpha}} = u_{\alpha} p^{\alpha} \Sigma(F - f) \tag{2.1}$$

where p^{α} is the four-momentum, x^{α} the space-time coordinates, Σ is a function of the local density and temperature having the meaning of the ratio between a collision frequency for a particle at zero speed, and c, $f = f(x^{\alpha}, p^{\alpha})$ is the distribution and F a Maxwell–Boltzmann distribution $F = \exp(A + B_{\alpha} p^{\alpha})$ varying from point to point in space-time in such a way that

$$\int u_{\alpha} p^{\alpha}(F - f)\, g_k \omega = 0 \tag{2.2}$$

where

$$g_{\beta} = p^{\beta}, \qquad g_4 = 1, \qquad \omega = dp^1 dp^2 dp^3/p \tag{2.3}$$

In spite of its linear appearance, Eq. (2.1) is highly nonlinear because of Eqs. (2.2) and (2.3) as well as of the dependence of Σ on density and temperature (see Refs. 6 and 7 for a discussion in the classical case). Letting

$$f = \Phi(1 + h) \tag{2.4}$$

where

$$\Phi = \exp(\bar{a} + \bar{b}_{\alpha} p^{\alpha}) \qquad (\bar{a}, \bar{b}_{\alpha} \text{ constants}) \tag{2.5}$$

and neglecting higher-order terms in h we obtain the linearized version of Eq. (2.1):

$$p^{\alpha} \frac{\partial h}{\partial x^{\alpha}} = \bar{\sigma} p^0 (a + b_{\alpha} p^{\alpha} - h) \tag{2.6}$$

where a and b^α are fields in space-time implicitly defined by

$$\int p^0 \Phi(a + b_\alpha p^\alpha - h)\, g_k \omega = 0 \qquad (k = 0, 1, 2, 3, 4) \tag{2.7}$$

and $\bar{\sigma}$ is the value of Σ in the unperturbed state. Here and above, as well as in the sequel, Greek indices run from 0 to 3 and the signature of space-time is taken to be $(-1, 1, 1, 1)$. Further, without any loss of generality, we have assumed $\bar{b}^\alpha = 0$ for $\alpha = 1, 2, 3$.

We shall now consider solutions depending on x^0 and x^1, but not upon x^2 and x^3, as appropriate for situations where the gas occupies a half space, say $x^1 > 0$, or the region between two parallel planes, say $|x^1| < d/2$. Then the equation to be solved can be written as follows:

$$\frac{\partial h}{\partial x^0} + \mu \frac{\partial h}{\partial x^4} = \bar{\sigma}(a + b_\alpha p^\alpha - h) \tag{2.8}$$

where

$$\mu = p^1/p^0 \tag{2.9}$$

is the speed of a particle in c units ($|\mu| < 1$). Again, in complete analogy with the classical case,[7] it is possible to separate the shear effects from the effects due to longitudinal disturbances. To this end it is sufficient to split h as follows:

$$h = h_1 + h_2 + h_3 \tag{2.10}$$

where

$$\begin{aligned} h_1 &= \Pi_1 h = \tfrac{1}{2}(I + P_2 P_3) h \\ h_2 &= \Pi_2 h = \tfrac{1}{4}(I + P_2)(I - P_3) h \\ h_3 &= \Pi_3 h = \tfrac{1}{4}(I + P_3)(I - P_2) h \end{aligned} \tag{2.11}$$

I is the identity and P_α denotes the operator reflecting the αth component of p^α (here $\alpha = 1, 2, 3$); thus, e.g., $P_3 f(p^1, p^2, p^3) = f(p^1, p^2, -p^3)$.

Note that $P_k^2 = I$, $P_h P_k = P_k P_h$ imply

$$\Pi_k \Pi_h = \Pi_k \delta_{hk}, \qquad \sum_{k=1}^{3} \Pi_k = I \tag{2.12}$$

i.e., in any Hilbert space of functions admitting P_k as symmetries, h is decomposed, according to (2.10), into three mutually orthogonal components.

Applying the orthogonal projectors Π_k to Eq. (2.8) we find

$$\frac{\partial h_1}{\partial x^0} + \mu \frac{\partial h_1}{\partial x^1} = \bar{\sigma}(a + b_0 p^0 + b_1 p^1 - h_1) \tag{2.13}$$

$$\frac{\partial h_2}{\partial x^0} + \mu \frac{\partial h_2}{\partial x^1} = \bar{\sigma}(b_2 p^2 - h_2) \tag{2.14}$$

$$\frac{\partial h_3}{\partial x^0} + \mu \frac{\partial h_3}{\partial x^1} = \bar{\sigma}(b_3 p^3 - h_3) \tag{2.15}$$

We remark now that, as a consequence of Eqs. (2.7), a and b_α are related to h_1, h_2, h_3 through integrals of the following form:

$$\begin{aligned} I &= \int h_1 p^0 \Phi \omega \\ I_0 &= \int h_1 (p^0)^2 \Phi \omega \\ I_1 &= \int h_1 p^0 p^1 \Phi \omega \\ I_2 &= \int h_2 p^0 p^2 \Phi \omega \\ I_3 &= \int h_3 p^0 p^3 \Phi \omega \end{aligned} \tag{2.16}$$

If one introduces as integration variables p^0, θ, μ, where

$$\begin{aligned} p^2 &= [p^{0^2}(1-\mu^2) - (mc)^2]^{1/2} \cos\theta \\ p^3 &= [p^{0^2}(1-\mu^2) - (mc)^2]^{1/2} \sin\theta \\ p^1 &= \mu p^0 \left(p^0 \geqslant \frac{mc}{(1-\mu^2)^{1/2}} \right) \end{aligned} \tag{2.17}$$

then

$$p^0 \omega = dp^1\, dp^2\, dp^3 = p^{0^2}\, d\mu\, dp^0\, d\theta \tag{2.18}$$

If one introduces the following scalar product for functions of p^0 and θ, depending parametrically on μ:

$$(f, g) = \int_{mc/(1-\mu^2)^{1/2}}^{\infty} \int_0^{2\pi} f g \Phi p^{0^2}\, dp^0\, d\theta \tag{2.19}$$

one easily sees that one can simplify the equations by splitting h_1 in a part along p^0, βY_0 ($\beta \equiv -b_0 > 0$), a part along 1, Y_1, and a remainder, h_{1R}, orthogonal to p^0 and 1 according to the scalar product defined in Eq. (2.19). While h_{1R} satisfies a simple partial differential equation:

$$\frac{\partial h_{1R}}{\partial x^0} + \mu \frac{\partial h_{1R}}{\partial x^1} = -\bar{\sigma} h_{1R} \tag{2.20}$$

Y_0 and Y_1 satisfy a system of the following form:

$$\frac{\partial Y_1}{\partial x^0} + \mu \frac{\partial Y_1}{\partial x^2} = \bar{\sigma}(a - Y_1) \tag{2.21}$$

$$\frac{\partial Y_0}{\partial x^0} + \mu \frac{\partial Y_0}{\partial x^1} = \bar{\sigma}[\beta^{-1}(b_1 \mu + b_0) - Y] \tag{2.22}$$

where, as said before,

$$h_1 = \beta Y_0 p^0 + Y_1 + h_{1R} \tag{2.23}$$

Y_0 and Y_1 depend, of course, on x^0, x^1, μ. The expressions of a, b_0, b_1 can be easily computed in terms of Y_0 and Y_1 but contain μ in a complicated way, typically taken to the power of 1/2 in the combination $(1 - \mu^2)^{1/2}$. If, however, we consider the ultrarelativistic limit when

$$\varepsilon = \beta mc = \frac{mc^2}{k_B T} \tag{2.24}$$

(T temperature of the unperturbed gas, k_B Boltzmann constant) goes to zero, we can easily express the terms a, b_0, and b_1. In fact if we let $p = \beta^{-1} w$, the variable w goes from $\varepsilon/(1 - \mu^2)^{1/2}$ to ∞ in Eq. (2.19); letting $\varepsilon \to 0$, the lower limit of integration disappears and a great semplification ensues. In fact, one obtains

$$a = \frac{1}{2} \int_{-1}^{1} Y_1 \, d\mu$$

$$b_0 = \frac{1}{2} \beta \int_{-1}^{1} Y_0 \, d\mu$$

$$b_1 = \frac{3}{8} \beta \int_{-1}^{1} Y_1 \mu \, d\mu + \frac{3}{2} \beta \int_{-1}^{1} Y_0 \mu \, d\mu \tag{2.25}$$

The last equation indicates that a better unknown in place of Y_0 is

$$Z = Y_0 + \tfrac{1}{4} Y_1 \tag{2.26}$$

In fact in terms of Z and Y_1 (that will be simply denoted by Y), one obtains

$$\frac{\partial Y}{\partial x^0} + \mu \frac{\partial Y}{\partial x^1} = \bar{\sigma} \left[\frac{1}{2} \int_{-1}^{1} Y(\mu')\, d\mu' - Y \right] \tag{2.27}$$

$$\frac{\partial Z}{\partial x^0} + \mu \frac{\partial Z}{\partial x^1} = \bar{\sigma} \left[\frac{1}{2} \int_{-1}^{1} Z(\mu')\, d\mu' + \frac{3}{2} \mu \int_{-1}^{1} \mu' Z(\mu')\, d\mu' - Z \right] \tag{2.28}$$

It is remarkable that the equations for Y and Z are very simple: Equation (2.27) is the one-speed transport equation with isotropic scattering well known from radiative transfer[13] and neutron transport[14] in the conservative case. Equation (2.28) is a similar equation containing a notable amount of anisotropy in the scattering; actually the coefficient 3 in front of the second integral is three times larger than the maximum value admissible for a scattering kernel in linear transport. We must remember, however, that we are not dealing with the linear Boltzmann equation, but with a linearized version of the nonlinear one; the resulting scattering kernel is by no means bound to be positive. Here, according to a rather standard nomenclature, the term *linear* applies to the transport equation for particles moving in a much denser host medium of equilibrium particles, while *linearized* applies to small perturbations from equilibrium when particles of the same species collide with each other.[7]

One may proceed in the same way with h_2 and h_3 by letting

$$h_2 = \beta p_2 Y_2, \qquad h_3 = \beta p_3 Y_3 \tag{2.29}$$

to find that Y_2 and Y_3 satisfy the same equation, having the form

$$\frac{\partial W}{\partial x^0} + \mu \frac{\partial W}{\partial x^1} = \bar{\sigma} \left[\frac{3}{4} \int_{-1}^{1} (1 - \mu'^2)\, W(\mu')\, d\mu' - W \right] \tag{2.30}$$

Equation (2.27) describes thermal waves, Eq. (2.30) shear waves, Eq. (2.28) sound waves; this is easily seen by the fact that Eq. (2.28) has two conservation equations. The form of the equations (2.27) and (2.28) is simpler than that of the classical limit ($\varepsilon \to \infty$); the latter leads, of course, to the traditional BGK model. Accordingly it is somewhat easy to study the solution of the half space problems in the ultrarelativistic limit, as will be indicated in the next few sections.

3. THE ELEMENTARY SOLUTIONS OF THE LAPLACE-TRANSFORMED EQUATION FOR Y

The general structure of the solutions of Eq. (2.27) was investigated by Bowden and Williams,[17] who extended a treatment due to Case[18] and applicable to the steady solutions. It is remarkable that, at the same time and independently, the same approach was used[19] to extend this author's approach to the classical BGK model,[20] that had been, in turn, inspired by Case's method.

In order to simplify the notation we introduce nondimensional time and space variables

$$t = x^0 \bar{\sigma}, \qquad x = x^1 \bar{\sigma} \tag{3.1}$$

so that Eq. (2.27) becomes

$$\frac{\partial Y}{\partial t} + \mu \frac{\partial Y}{\partial x} + Y = \frac{1}{2} \int_{-1}^{1} Y(x, t, \mu')\, d\mu' \tag{3.2}$$

Let us take the Laplace transform of this equation. Without loss of generality a zero initial value for Y will be assumed; in fact from the properties of the resulting homogeneous equation, a particular solution of the inhomogeneous one, which would result from a nonzero initial condition, can always be constructed. Thus in every case we are reduced to treat the homogeneous equation:

$$(s+1)\,\tilde{Y} + \mu \frac{\partial \tilde{Y}}{\partial x} = \frac{1}{2} \int_{-1}^{1} \tilde{Y}(x, s, \mu')\, d\mu' \tag{3.3}$$

where $\tilde{Y}$ denote the Laplace transform of Y. The same equation (with $s = i\omega$) is obtained when studying steady oscillations. It is to be noted that our $s+1$ corresponds to s of Ref. 17. Also Ref. 17 discusses a slightly more general case containing a parameter, resulting equal to unity in our case. The form of Eq. (3.3) suggests looking for separated variable solutions of the form

$$\tilde{Y}(x, s, \mu) = e^{-(s+1)x/\nu} f_\nu(\mu, s) \tag{3.4}$$

where f satisfies

$$\left(1 - \frac{\mu}{\nu}\right) f_\nu(\mu, s) = \frac{1}{2(s+1)} \int_{-1}^{1} f_\nu(\mu', s)\, d\mu' \tag{3.5}$$

These solutions are usually called elementary solutions, according to the terminology used by Case.[18]

The right hand side of Eq. (3.5) does not depend on μ and can be normalized to unity. Accordingly, we are led to a division problem, typical in the theory of generalized functions: if the factor $(1 - \mu/\nu)$ is nonzero, i.e., $\nu \notin (-1, 1)$, $f_\nu(\mu)$ is an ordinary function given by

$$f_\nu(\mu, s) = \frac{\nu}{\nu - \mu} \tag{3.6}$$

with the normalization condition:

$$\frac{1}{2(s+1)}\int_{-1}^{1}\frac{\nu}{\nu-\mu}\,d\mu=1 \tag{3.7}$$

or

$$s+1=\nu\tanh^{-1}(1/\nu) \tag{3.8}$$

where the branch of $\tanh^{-1}(1/\nu)$ to be chosen is such that it is zero when the argument is zero (i.e., $\nu\to\infty$) and is continuous in the complex plane cut along the real interval $(-1, 1)$.

If, on the contrary, $\nu\in(-1, 1)$, $f_\nu(\mu)$ must be considered to be a generalized function or distribution, and Eq. (3.5) gives

$$f_\nu(\mu, s)=P\frac{\nu}{\nu-\mu}+\lambda(\nu, s)\,\delta(\nu-\mu) \qquad [\nu\in(-1, 1)] \tag{3.9}$$

where

$$\lambda(\nu, s)=2(s+1)-\nu\log\frac{1+\nu}{1-\nu} \tag{3.10}$$

and the symbol P means the Cauchy principal value. Eq. (3.9) gives the generalized eigensolutions corresponding to the continuous spectrum $(-1<\nu<1)$.

The next step is to study the values of ν for which Eq. (3.8) is satisfied, i.e., for any given value of s the zeros of the function

$$\Omega(\nu, s)=s+1-\nu\tanh^{-1}(1/\nu) \tag{3.11}$$

This function is continuous in the complex plane of ν cut along the real interval $(-1, 1)$. In the limiting case when ν tends to a value on the cut, the discrete spectrum merges into the continuous spectrum and the zeros satisfy to

$$s+1=\frac{\nu}{2}\log\frac{1+\nu}{1-\nu}\pm\frac{\pi}{2}i\nu \qquad [\nu\in(-1, 1)] \tag{3.12}$$

This equation is satisfied on a curve C in the complex plane of the variable s lying all at the right of the straight line $\operatorname{Re} s=-1$ tangent to the curve at $s=1$, and between the asymptotes $\operatorname{Im} s=\pm\pi/2$ (see, e.g., Ref. 14, p. 177).

Using this curve and the principle of the argument it is easy to see that Eq. (3.8) has no solutions if s is at the left of the curve C ($s\in L$), two opposite solutions $\pm v_0$ [not lying in the interval $(-1, 1)$] when s is at the right of C ($s \in R$). When s is on C the two opposite roots are $v = \pm 2/\pi \operatorname{Im} s$ and are on the interval $(-1, 1)$. The corresponding eigensolutions will be denoted by $f_{\pm}(\mu, s)$.

It is easy to prove, following standard methods,[18,19] that generalized eigenfunctions corresponding to different values of v (in either the discrete or continuous spectrum) are orthogonal with respect to the (indefinite) weight μ:

$$\int_{-1}^{1} \mu f_v(\mu, s) f_{v'}(\mu, s)\, d\mu = 0, \qquad v \neq v' \tag{3.13}$$

If we include the case $v = v'$ we find

$$N_{\pm}(s) \equiv \int_{-1}^{1} \mu [f_{\pm}(\mu, s)]^2\, d\mu = \pm 2 v_0 \left(\frac{v_0}{v_0^2 - 1} - s \right), \qquad s \in R \tag{3.14}$$

for the discrete spectrum and

$$\int_{-1}^{1} \mu f_v(\mu, s) f_{v'}(\mu, s)\, d\mu = v \{ [\lambda(v, s)]^2 + \pi^2 v^2 \}\, \delta(v - v') \tag{3.15}$$

for the continuous one. This is a symbolic formula whose meaning is the following. If we expand a given function as an integral of the eigenfunctions of the continuous set (in the sense of distributions), then in order to compute the coefficient of the expansion $A(v)$, we can use Eq. (3.15) formally, i.e., exchanging freely the order of integrations to obtain the correct result. A rigorous treatment would involve use of the Poincaré–Bertrand formula.[21] For a more complete discussion on this point see, e.g., Ref. 14, pp. 69–71.

The set of generalized functions $\{f_v(\mu)\colon v(-1, 1) \cup \{\pm v_0\}\}$ are a complete set for sufficiently well-behaved functions $Y(\mu)$ defined on $(-1, 1)$. A sufficient condition is that $\mu Y(\mu)$ obey an H condition on $(-1, 1)$.[21] This means that Y is Hölderian in any closed subinterval of $(-1, 1)$ and is Hölderian on the closed interval $(-1, 1)$ when multiplied by $(1 - \mu^2)^{\delta}$ $(0 < \delta < 1)$. The proof is standard.[18,20]

There are more general completeness theorems for subintervals (μ_1, μ_2) where the set needed is $\{f_v(\mu)\colon v(\mu_1, \mu_2)\}$. The most interesting case refers to the interval $(0, 1)$.

In this case it is important to quote the rule to compute the coefficients of the representation of $Y(\mu)$ in terms of the set $\{f_v(\mu)\colon v \in (0, 1)\}$. When $s \in L$ one has

$$Y(\mu) = \int_{0}^{1} A(v, s) f_v(\mu, s)\, dv \qquad (0 < \mu < 1) \tag{3.16}$$

$$A(\nu, s) = \frac{\lambda(\nu, s)}{[\lambda(\nu, s)]^2 + \pi^2 \nu^2} Y(\nu) - \frac{1}{X_A^-(\nu, s)[\lambda(\nu, s) + \pi i \nu]} \times P \int_0^1 \frac{\mu X_A^-(\mu, s)\, Y(\mu)}{\lambda(\mu, s) - \pi i \mu} \frac{d\mu}{\mu - \nu} \tag{3.17}$$

Here $X_A^-(\mu, s) = \lim_{\varepsilon \to 0} X(\mu - i\varepsilon, s)$ and the following function of the complex variable z has been introduced:

$$x_A(z, s) = \exp\left[\frac{1}{2\pi i} \int_0^1 \frac{G(\mu, s)}{\mu - z} d\mu\right] \tag{3.18}$$

where

$$G(\mu, s) = \log\left[\frac{\lambda(\mu, s) + \pi i \mu}{\lambda(\mu, s) - \pi i \mu}\right] \tag{3.19}$$

Here the branch of the logarithm is chosen in such a way that $G(\mu, s) \to 0$ when $\mu \to 1$.

When $s \in R$ we have

$$Y(\mu) = B_0 f_+(\mu, s) + \int_0^1 B(\nu, s) f_\nu(\mu, s)\, d\nu \qquad (0 < \mu < 1) \tag{3.20}$$

where

$$B_0 = \int_0^1 \frac{\mu Y(\mu) X_B^-}{\lambda(\mu, s) - \pi i \mu} d\mu \Bigg/ \int_0^1 \frac{\mu X_B^- f_+(\mu)}{\lambda(\mu, s) - \pi i \mu} d\mu \tag{3.21}$$

and

$$B(\nu, s) = \frac{\lambda(\nu, s)}{[\lambda(\nu, s)]^2 + \pi^2 \nu^2} [Y(\nu) - B_0 f_+(\nu)] - \frac{1}{X_B^-(\nu, s)[\lambda(\nu, s) + \pi i \nu]} P \int \frac{\mu X_B^-(\mu, s)[Y(\mu) - B_0 f_+(\mu)]}{[\lambda(\mu, s) - \pi i \mu](\mu - \nu)} d\mu \tag{3.22}$$

Here $X_B^-(\mu, s) = \lim_{\varepsilon \to 0+} X_B(\mu - i\varepsilon, s)$ and

$$X_B(z, s) = z^{-1} \exp\left[\frac{1}{2\pi i} \int_0^1 \frac{G(\mu, s)}{\mu - z} d\mu\right] \tag{3.23}$$

These expressions are obtained by solving in the usual way[21,11] the singular equations (3.16) and (3.20).

We note that X_A and X_B satisfy certain identities such as

$$X_A(z, s)\, X_A(-z, s) = \frac{\Omega(z, s)}{s} \tag{3.24}$$

$$X_B(z, s)\, X_B(-z, s) = \frac{\Omega(z, s)}{(v^2 - z^2)s} \tag{3.25}$$

where the function Ω is defined by Eq. (3.11).

These identities can be used to put in real form Eqs. (3.17), (3.21), and (3.22). The proof of these identities is straightforward (see, e.g., Ref. 11).

4. APPLICATION TO THE PROPAGATION OF A SHARP PULSE

In this section the general method developed in the previous section is applied to a typical problem, i.e., that of propagation of an impulsive change of temperature at the boundary of a half space $x > 0$ filled with gas. This means that we solve Eq. (3.2) in $x > 0$ with the following initial and boundary conditions:

$$Y(0, t, \mu) = 1 \qquad (\mu > 0) \tag{4.1}$$

$$Y(x, 0, \mu) = 0 \tag{4.2}$$

Further Y must be bounded at infinity.

While the classical case for a transversal velocity change has been studied in Ref. 19, the problem of a temperature change has not been treated, to this author's knowledge.

Introducing the Laplace transform of Y leads to Eq. (3.3) with the boundary condition

$$\tilde{Y}(0, s, \mu) = 1/s \qquad (\mu > 0) \tag{4.3}$$

According to the general method developed in Section 3, the solution $\tilde{Y}$ when $s \in L$ can be written as folows:

$$\tilde{Y}(x, s, \mu) = \int_0^1 A(v)\, e^{-(s+1)x/v} f_v(\mu, s)\, dv \tag{4.4}$$

This expression holds in the half-plane $\operatorname{Re} s > -1$; $A(v)$ has be taken different from zero only for $v > 0$, because of the condition at infinity. $A(v)$ is now evaluated by matching the boundary condition (4.3):

$$\frac{1}{s} = \int_0^1 A(v, s)\, f_v(\mu, s)\, dv \tag{4.5}$$

According to the general formula (3.17) we can evaluate

$$A(v, s) = \frac{1}{s} \frac{1}{X_A^-(v)[\lambda(v, s) + \pi i v]} \tag{4.6}$$

where use has been made of the identity

$$X_A(z, s) = 1 + \int_0^1 \frac{\mu X_A^-(\mu, s)}{\lambda(\mu, s) - \pi i \mu} \frac{d\mu}{\mu - z} \tag{4.7}$$

which follows by means of the Plemelj formulas, in the way indicated in Ref. 11. The expression of $A(v, s)$ can be put in real form by means of Eq. (3.24) and the solution $\tilde{Y}$ is given by

$$\tilde{Y}(x, s, \mu) = 2 \int_0^1 \frac{X_A(-v, s) f_v(\mu, s)}{[\lambda(v, s)]^2 + \pi^2 v^2} e^{-(s+1)x/v} \, dv \qquad (s \in L) \tag{4.8}$$

When $s \in R$ we have

$$\tilde{Y}(x, s, \mu) = B_0 f_+(\mu, s) e^{-(s+1)x/v_0} + \int_0^1 B(v) e^{-(s+1)x/v} f_v(\mu, s) \tag{4.9}$$

where v_0 must be selected between the two possible values according to the condition at infinity; in fact v_0 will be fixed by

$$\text{Re}\left(\frac{s+1}{v}\right) \geqslant 0 \tag{4.10}$$

One can always find one and only one such a v for each $s \in R/(-1, 0)$; in fact if $s \in (-1, 0)$ the two values of v satisfying Eq. (3.8) are purely imaginary and satisfy the condition given by Eq. (4.10). Accordingly, the interval $(-1, 0)$ is excluded for the moment from our consideration.

B_0 and $B(v)$ are easily evaluated through the boundary condition (4.3) and the general formulas (3.21) and found to be

$$B_0 = \frac{1}{s v_0 X_B(v_0, s)} \tag{4.11}$$

$$B(v) = -\frac{1}{s(v_0 - v) X_B^-(v)[\lambda(v, s) + \pi i v]} \tag{4.12}$$

where use has been made of the identity

$$X_B(z, s) = \int_0^1 \frac{\mu X_B^-(\mu, s)}{\lambda(\mu, s) - \pi i \mu} \frac{d\mu}{\mu - z} \tag{4.13}$$

analogous to Eq. (4.7).

Accordingly,

$$\tilde{Y}(x, s, \mu) = \frac{e^{-(s+1)x/\nu_0}}{s(\nu_0 - \mu)\, X_B(\nu_0)} - 2 \int_0^1 \frac{(\nu_0 + \nu)\, X_B(-\nu, s)\, f_\nu(\mu, s)}{[\lambda(\nu, s)]^2 + \pi^2 \nu^2}\, e^{-(s+1)x/\nu}\, d\nu \qquad (s \in R) \quad (4.15)$$

Equation (4.8) and (4.14) give two different expressions for $\tilde{Y}$ according to whether $s \in L$ or $s \in R$. However, there is no singularity of $\tilde{Y}$, as function of s, on the curve C which separates L from R. As a matter of fact Eqs. (4.8) and (4.14) are the analytic continuation of each other through C. To see this, one has first to note that when s crosses C at $s = \bar{s}$, $\nu_0(s) \to \bar{\nu}$, where $\bar{\nu} \in (0, 1)$ is the real positive value corresponding to $\bar{s} \in C$ in the parametric representation (3.12). Then the following relation,

$$X_A(z, \bar{s}) = -(\bar{\nu} - z)\, X_B(z, \bar{s}) \qquad (\bar{s} \in C) \qquad (4.15)$$

must be used. Here, of course, $X_A(z, \bar{s})$ and $X_B(z, \bar{s})$ are the limits from L and R, respectively, of $X_A(z, s)$ and $X_B(z, s)$. Equation (4.15) can be obtained directly from Eqs. (3.18) and (3.23) or by the following indirect argument. Equations (3.24) and (3.25) give

$$X_A(z, \bar{s})\, X_A(-z, \bar{s}) = X_B(z, \bar{s})\, X_B(-z, \bar{s})(\bar{\nu}^2 - z^2) \qquad (\bar{s} \in C) \quad (4.16)$$

because $\Omega(z, s)$ is continuous through C. Hence

$$\frac{X_A(z, \bar{s})}{X_B(z, \bar{s})(\bar{\nu} - z)} = \frac{X_B(-z, \bar{s})(\bar{\nu} + z)}{X_A(-z, \bar{s})} \qquad (\bar{s} \in C) \qquad (4.17)$$

The function of z appearing in the left-hand side of this equation is analytic in $\operatorname{Re} z < 0$ except, at most, $z = 0$. The function defined by either side of Eq. (4.17) is, accordingly, analytic everywhere. Further this function is bounded and hence reduces to a constant because of Liouville's theorem. The common value of both sides of Eq. (4.17) is -1, as is seen by letting $z \to \infty$. Hence Eq. (4.15) follows.

Having established this, it is a simple matter to see that Eqs. (4.14) and (4.7) coincide on C. (Note that $[\lambda(\nu, s)]^2 + \pi^2 \nu^2$ has a zero at $\nu = \bar{\nu}$ when $s = \bar{s}$.)

We remark that it is useful to discuss the integral of $Y(x, t, \mu)$,

$$q(x, t) = \frac{1}{2} \int_{-1}^{1} Y(x, t, \mu)\, d\mu \qquad (4.18)$$

which has a simple physical significance (it is related to the temperature) rather then dealing with Y itself. The Laplace transform of q, $\tilde{q}$, is easily obtained by integrating Eqs. (4.8) and (4.14):

$$\tilde{q}(x, s) = 2(s+1) \int_0^1 \frac{X_A(-v, s)}{[\lambda(v, s)]^2 + \pi^2 v^2} e^{-(s+1)x/v} \, dv \qquad (s \in L) \quad (4.19)$$

$$\tilde{q}(x, s) = \frac{(s+1) e^{-(s+1)x/v_0}}{s v_0 X_B(v_0)} - 2(s+1) \int_0^1 \frac{(v_0 v) X_B(-v, s) e^{-(s+1)x/v}}{[\lambda(v, s)]^2 + \pi^2 v^2} \, dv \quad (s \in R) \quad (4.20)$$

We can now discuss the singularities of $\tilde{q}(x, s)$ as a function of s; the straight line $\operatorname{Re} s = -1$ appears to be a natural boundary for the analytic continuation of $\tilde{q}$, because the integral (4.20) does not exist when $\operatorname{Re} s < -1$. Consequently $\tilde{q}(x, s)$ is defined in the half plane $\operatorname{Re}(s+1) > 0$ with a cut along the interval $(-1, 0)$ of the real axis. In fact we have not defined $\tilde{q}(x, s)$ on this interval; moreover, when approaching it from below, it is seen that $v_0(s)$ and consequently $\tilde{q}(x, s)$, suffer a discontinuity, because of Eq. (4.10). We finally remark that $\tilde{q}(x, s)$ behaves as s^{-1} when $s \to 0$; in fact $v_0 X_B(v_0) \to 1$ when $s \to 0$ ($v_0 \to \infty$). The pole has residue 1.

We can now invert the Laplace transform to obtain

$$q(x, t) = \frac{1}{2\pi i} \int_{a+i\infty}^{a+i\infty} e^{st} \tilde{q}(x, s) \, ds \qquad (a > 0) \qquad (4.21)$$

where $\tilde{q}(x, s)$ is given by Eq. (4.19) or (4.20) according to $s \in L$ or $s \in R$. Because of the singularities of $\tilde{q}(x, s)$, it is seen that such a path can be deformed to a path indented on the segment $(-1, 0)$ of the real axis and along the vertical line $\operatorname{Re}(s+1) = 0$. Accordingly,

$$q(x, t) = 1 + \frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} e^{st} \tilde{q}(x, s) \, ds - \frac{1}{2\pi i} \int_{-1}^{0} e^{st} \Delta(x, s) \, ds \qquad (4.22)$$

where $\Delta(x, s)$ is the jump of $\tilde{q}(x, s)$ through the segment $(-1, 0)$ of the real axis. This formula is useful to study the long-time behavior of the solution ($t \gg 1$, as well as $t \gg x$). It is not hard to see that a sort of diffusive process is taking place [note that $v \simeq (3s)^{-1/2}$ when $s \to 0$]. For $t \to \infty$, $q \to 1$, i.e., the entire space has been invaded by the disturbance.

For short times, however, the situation is quite different. In particular, if $t < x$ (i.e., we consider a point located outside the light cone), we can evaluate the integral in Eq. (4.21) by closing the path in the half plane $\operatorname{Re} s > a > 0$; in fact $|\tilde{q}(x, s)|$ is bounded by $A(\sigma) e^{-\sigma x}$, where $A(\sigma)$ is, at most, a function of polynomial growth, when $\sigma \equiv \operatorname{Re} s \to \infty$. $\tilde{q}(x, s)$ does not possess singularities for $\operatorname{Re} s > 0$; hence

$$q(x, t) \equiv 0 \qquad (t < x) \tag{4.23}$$

In other words the wave front produced by the impulsive change at $x = 0$ propagates at a speed not larger than c.

5. APPLICATION TO THE PROPAGATION OF FORCED WAVES IN A HALF SPACE

In this section we consider a problem strictly related to the one treated in the previous section. We consider a gas filling a half space bounded by an infinite plane, whose temperature oscillates with a fixed frequency ω. The system is assumed to be in a steady state when the transients have disappeared. Accordingly the solution of Eq. (3.2) will have the form $Y(x, t, \mu) = e^{i\omega t} \hat{Y}(x, t, \mu)$, where $\hat{Y}$ satisfies

$$i\omega \hat{Y} + \mu \frac{\partial \hat{Y}}{\partial x} + \hat{Y} = \frac{1}{2} \int_{-1}^{1} \hat{Y}(x, \mu')\, d\mu' \tag{5.1}$$

with the boundary condition

$$\hat{Y} = 1 \qquad (\mu > 0) \tag{5.2}$$

where the amplitude of the vibration, without any loss of generality, has been taken to be unity. As before, the solution is required to be bounded at infinity. It is easy to remark that

$$\hat{Y}(x, \mu) = i\omega \hat{Y}(x, i\omega, \mu) \tag{5.3}$$

where $\hat{Y}(x, s, \mu)$ was found in the previous section by solving Eq. (3.3) with the boundary condition given by Eq. (4.3). Accordingly the quantity defined in Eq. (4.18) can be computed as follows:

$$q(x, t) = e^{i\omega t} \hat{q}(x) \tag{5.4}$$

where

$$\hat{q}(x) = 2i\omega(1 + i\omega) \int_0^1 \frac{X_A(-v, i\omega)}{[\lambda(v, i\omega)]^2 + \pi^2 v^2} e^{-(1 + i\omega)x/v}\, dv \qquad (|\omega| > \omega_c) \tag{5.5}$$

$$\hat{q}(x) = \frac{(1 + i\omega) e^{-(1 + i\omega)x/v_0}}{v_0 x_B(v_0)} - 2i\omega(1 + i\omega) \int_0^1 \frac{(v_0 + v) x_B(-v, i\omega) e^{-(1 + i\omega)x/v}}{[\lambda(v, i\omega)]^2 + \pi^2 v^2} dv \qquad (|\omega| < \omega_c) \tag{5.6}$$

Here X_A and X_B are the functions defined in Section 3,

$$\omega_c \simeq 1.199678640 \tag{5.7}$$

and ν_0 is such that

$$1 + i\omega = \nu_0 \tanh(1/\nu_0) \tag{5.8}$$

This equation has solution if and only if $|\omega| < \omega_c$ (see Ref. 4); this value, of course, is equal to the ordinate of the intersection of the curve C with the imaginary axis of the complex s plane.

In Ref. 4 the dispersion relation (5.8) was analytically continued beyond the critical value to yield solutions for $|\omega| > \omega_c$. Here, in analogy with Ref. 11, we can give the physical meaning to this procedure. As a matter of fact we can consider the complex plane of the variable and deform the integration path in Eq. (5.5), provided we keep the end points fixed and add the contribution from any pole of the integrand between the old and the integration path. Now it is easily seen that, at least for frequencies larger but still close to ω_c, there is one such pole, ν_0, which satisfies

$$\lambda(\nu_0, i\omega) - \pi i \nu_0 = 0 \tag{5.9}$$

provided we deform the path into the quadrant $\operatorname{Re}\nu > 0$, $\operatorname{Im}\nu < 0$. Equation (5.9) is the analytic continuation of the dispersion relation $\Omega(\nu_0, i\omega) = 0$ for $\omega > \omega_c$. Accordingly, Eq. (5.5) takes on a shape similar to Eq. (5.6) except for the slight change in the integration path. Thus we uncover a discrete term even when no discrete spectrum exists! This term will have a physical meaning if it dominates the contribution from the remaining integral. This is certainly not true at all distances even at $\omega \simeq \omega_c$; in fact the less damped contribution from the integral in Eq. (5.6) decays as e^{-x}, while for $\omega \simeq \omega_c$ and hence $\nu_0^{-1} \sim \pi/(2\omega_c) \simeq 1.31$, the discrete term decays as $\exp[-(1.31)x]$. Hence the integral dominates over the discrete term at large distances.

Yet, the discrete term might be significantly larger at small and intermediate distances. We cannot expect, however, this to be true at all the frequencies, because the root ν_0 of Eq. (5.9) will tend to have a large real part ($\nu_0 \simeq \omega/\pi$) and hence to uncover it, it will be necessary to deform the path in such a way as to produce large contributions from the integral term as well.

In fact, we can consider the analytic continuation rather meaningless when $\operatorname{Re}\nu_0 \simeq 1$. Taking into account the fact that $\nu_0 \simeq 2\omega/\pi$ for $\omega \simeq \omega_c$ and $\nu_0 \simeq \omega/\pi$ for $\omega \to \infty$, we can estimate that the analytic continuation will loose any meaning for frequencies larger than a second critical value between 1.57 and 3.14. This appears to be in a roughly good agreement with the numerical results of Ref. 4, indicating that for $\omega > 2.5$ the analytically continued dispersion relation gives phase speeds larger than the speed of light, in disagreement, not only with physical intuition, but also with the general property proved in Ref. 1.

6. THE ELEMENTARY SOLUTIONS OF THE LAPLACE-TRANSFORMED EQUATIONS FOR *Z* AND *W*

A treatment similar to the one expounded in the previous sections can be applied to Eqs. (2.28) and (2.30). The second of these equations is dealt with in exactly the same way used for Eq. (2.27). In fact, written in nondimensional space and time variables and Laplace transformed, Eq. (2.30) takes on the form

$$(s+1)\tilde{W} + \mu \frac{\partial \tilde{W}}{\partial x} = \frac{3}{4}\int_{-1}^{1} \tilde{W}(x, s, \mu')(1-\mu'^2)\, d\mu' \tag{6.1}$$

The separate variable solutions are again given by the product of an exponential in x times $f_\nu(\mu, s)$, a function given by Eq. (3.6) or a generalized function given by Eq. (3.9). The dispersion relation, Eq. (3.8), and the expression of $\lambda(\nu, s)$, Eq. (3.10) change, however, into

$$s+1 = (3/2)\nu(1-\nu^2)\tanh^{-1}(1/\nu) + (3/2)\nu^2 \tag{6.2}$$

$$\lambda(\nu, s) = \left[\frac{4}{3}(s+1) - 2\nu^2\right]\frac{1}{1-\nu^2} - \nu \log\frac{1+\nu}{1-\nu} \tag{6.3}$$

As a consequence the curve bounding the region of existence of the continuous spectrum is given by

$$s+1 = \frac{3}{2}\nu^2 + \frac{3}{4}\nu(1-\nu^2)\log\frac{1+\nu}{1-\nu} \pm \frac{3\pi}{4} i\nu(1-\nu^2) \tag{6.4}$$

In this case the region inside the curve is heart-shaped and occupies a finite part of the complex plane. The curve meets the real axis at $s = 0.5$ and $s = -1$, the imaginary axis at $\operatorname{Im} s \simeq 0.9$.

The treatment of general properties and applications parallels the one given for Eq. (2.27).

Slightly more complicated is the case of Eq. (2.28). After putting it in nondimensional form, Laplace transforming and looking for separated variable solutions, one is faced with

$$(s+1)\left(1-\frac{\mu}{\nu}\right) f_\nu(\mu, s) = \frac{1}{2}\int_{-1}^{1} f_\nu(\mu', s)\, d\mu' + \frac{3}{2}\mu\int_{-1}^{1} f_\nu(\mu', s)\mu'\, d\mu' \tag{6.5}$$

The new fact is given by the presence of two integrals in the left-hand side. Similar situations were considered in Ref. 11. The best way to deal with Eq. (6.5) is to remark that integrating both sides of Eq. (6.5) with respect to μ from -1 to 1 gives

$$s\int_{-1}^{1} f_\nu(\mu, s) - \frac{s+1}{\nu}\int_{-1}^{1} \mu f_\nu(\mu, s) = 0 \tag{6.6}$$

Hence we can express one of the integrals in terms of the other and rewrite Eq. (6.5) in the form

$$\left(1 - \frac{\mu}{\nu}\right) f_\nu(\mu, s) = \frac{1}{2}\left[\frac{1}{s+1} + \frac{3s\mu\nu}{(s+1)^2}\right]\int_{-1}^{1} f_\nu(\mu, s) \tag{6.7}$$

The dispersion relation now becomes

$$\left[\frac{1}{s+1} + \frac{3s\nu^2}{(s+1)^2}\right]\tanh^{-1}\left(\frac{1}{\nu}\right) - \frac{3s\nu^2}{(s+1)^2} = 1 \tag{6.8}$$

The boundary of the region where a solution to this dispersion relation exists is given by

$$s = \frac{-B \pm (B^2 - 4C)^{1/2}}{2} \tag{6.9}$$

where

$$B = 1 + (1 + 3\nu^2)\left(1 + \frac{\nu}{2}\log\frac{1-\nu}{1+\nu} \pm i\frac{\nu}{2}\pi\right) \tag{6.10}$$

$$C = 1 + \frac{\nu}{2}\log\frac{1-\nu}{1+\nu} \pm i\frac{\nu}{2}\pi \tag{6.11}$$

The curve has a double point at $s = -1$ $(\nu = 0)$. This point separates the curve into two parts: a closed path C_1 lying in the strip $-1 \leqslant \operatorname{Re} s \leqslant 1/4$, and a curve C_2 going to infinity for large values of Re s, in a way similar to the curve C met in Section 3. It is to be expected that no discrete spectrum exists when $s \in R$, where R is the region at the right of C_2. On the other hand, two discrete eigenvalues, opposite of each other, should exist when $s \in R$, where R_2 is R deprived of the region R_1 bounded by C_1, and four eigenvalues when $s \in R_1$. A detailed study of the shape of C_1 and C_2, as well as of the roots of the dispersion relation (6.8) is beyond the scope of the present paper. It can be conjectured, however, that many of the results obtained in the previous sections are valid for Eq. (2.28). This equation, however, seems to deserve further consideration, because it is capable of describing the propagation of sound waves in a very simple way.

7. CONCLUDING REMARKS

We have shown that the kinetic theory of an ultrarelativistic gas can be reduced to the solution of three simple uncoupled equations, provided we adopt a linearized single relaxation model and consider $(1+1)$-dimensional problems. One of these equations has been studied in detail, taking advantage of the fact that many results are already available. The other two equations, and in particular the equation describing sound waves, seem to deserve further consideration.

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